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| Title | Unconditional well-posedness of fifth order KdV type equations with periodic boundary condition (Harmonic Analysis and Nonlinear Partial Differential Equations) |
| Author(s) | Kato, Takamori |
| Citation | 数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2018), B70: 105-129 |
| Issue Date | 2018-04 |
| URL | http://hdl.handle.net/2433/243753 |
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| Type | Departmental Bulletin Paper |
| Textversion | publisher |

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**B70** (2018), 105–129

# Unconditional well-posedness of fifth order KdV type equations with periodic boundary condition

By

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## Abstract

This paper is announcement of the result obtained in [8] by Tsugawa and the author. We study the well-posedness of the Cauchy problem of the fifth order KdV type equations on $\mathbb{T}$. We show the local well-posedness and unconditional uniqueness in $H^s(\mathbb{T})$ for $s \geq 3/2$. The main idea of the proof is using the conserved quantities to cancel the resonant parts with a loss of derivatives and applying the normal form reduction to the non-resonant parts to recover derivatives.

## § 1. Introduction

This paper is announcement of the result obtained in [8] by Tsugawa and the author. We consider the Cauchy problem of the fifth order KdV type equation on one dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

$$\partial_t u + \partial_x^5 u + \alpha \partial_x (\partial_x u)^2 + \beta \partial_x (u \partial_x^2 u) + 10\gamma \partial_x (u^3) = 0, \tag{1.1}$$

$$u(0, \cdot) = \varphi(\cdot) \in H^s(\mathbb{T}) \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $u : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ and $\varphi : \mathbb{T} \to \mathbb{R}$. We observe some conserved quantities. Put

$$E_0(u(t)) := \frac{1}{2\pi} \int_{\mathbb{T}} u(t)\ dx, \quad E_1(u(t)) := \frac{1}{2\pi} \int_{\mathbb{T}} u^2(t)\ dx,$$
$$E_3(u(t)) := \frac{1}{4\pi} \int_{\mathbb{T}} \big(\partial_x^2 u(t)\big)^2 + 5\gamma u^4(t) - \beta u(t) \big(\partial_x u(t)\big)^2\ dx$$

Received October 9, 2017. Revised February 26, 2018.
2010 Mathematics Subject Classification(s): 35Q53, 37K10, 35A01, 35B65.
*Key Words*: *KdV hierarchy, Fifth order, Normal form, Well-posedness, Cauchy problem, Low regularity, Unconditional.*
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We have conservation laws $E_1(u(t)) = E_1(\varphi)$ and $E_3(u(t)) = E_3(\varphi)$ in the formal sense when the assumption $\beta = 2\alpha$ holds. Moreover, $E_0(u(t)) = E_0(\varphi)$ holds without any assumption. In this case $\beta = 2\alpha$, (1.1) can be regarded as the Hamiltonian PDE $\partial_t u = -\partial_x \nabla_u H(u)$ with $H(u) := E_3(u)$. In this case $(\alpha, \beta, \gamma) = (\pm 5, \pm 10, 1)$, (1.1) is called the fifth order KdV equation, which is completely integrable and the second equation in the KdV hierarchy discovered by Lax. We remark that it is certain that the inverse scattering theory is not applicable because (1.1) is no longer complete integrability.

In [2], Benny introduced the following equation to describe interactions between short and long waves:

$$\partial_t u + \partial_x^5 u = \frac{1}{2}\partial_x(\partial_x u)^2 + \partial_x(u\partial_x^2 u).$$

There are many well-posedness results for the KdV equations on $\mathbb{T}$,

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0. \tag{1.3}$$

However, known results for the fifth order KdV type equations are few. This is because the strong singularity in the nonlinear terms make the problem difficult. To overcome this difficulty, we obtain the main result as follows:

**Theorem 1.1.** *Let $s \geq 3/2$ and $\beta = 2\alpha$. Then, for any $\varphi \in H^s(\mathbb{T})$, there exists $T = T(\|\varphi\|_{H^{3/2}}) > 0$ such that there exists a unique solution $u \in C([-T, T] : H^s(\mathbb{T}))$ to* (1.1)–(1.2)*. Moreover the solution map, $H^s(\mathbb{T}) \ni \varphi \mapsto u \in C([-T, T] : H^s(\mathbb{T}))$ is continuous.*

When $\beta = 2\alpha$ holds, we have the following global result as a corollary of Theorem 1.1 thanks to the conservation of $E_3$.

**Corollary 1.2.** *We assume the same assumption in Theorem 1.1 and $\varphi \in H^2(\mathbb{T})$. Then, the solution obtained in Theorem 1.1 can be extended to the solution on $t \in (-\infty, \infty)$.*

*Remark* 1.1. Theorem 1.1 and Corollary 1.2 claim that unconditional uniqueness holds. Unconditional uniqueness means that uniqueness holds in $C([0, T] : H^s(\mathbb{T}))$ without intersecting with any auxiliary function space. This is a concept of uniqueness which does not depend on how solutions are constructed. In the context of unconditional well-posedness, refer to [7].

Main idea of the proof of Theorem 1.1 is how to deal with a loss of derivatives. Key ingredients are how to cancel the resonant parts with a loss of the derivatives and how to recover derivative losses in the non-resonant parts. The resonant parts does not have any smoothing effect because the oscillation effects are canceled. (We give the rigorous definition of the resonant parts and the non-resonant parts in the latter part of this section.)

When $x \in \mathbb{R}$, the following linear local smoothing is due to Kenig, Ponce and Vega [10]:

$$\|\partial_x^j e^{-t\partial_x^{2j+1}} \varphi\|_{L_x^\infty L_t^2} \leq C \|\varphi\|_{L^2}$$

We observe that this smoothing effects enables us to recover two derivatives when $j = 2$. However, when $x \in \mathbb{T}$, the linear part does not have any smoothing effects.

Now we briefly go over recent results on the well-posedness theory of the KdV equation (1.3) on $\mathbb{T}$. Bourgain [3] proved the local well-posedness of (1.3) in $L^2(\mathbb{T})$, which was improved to LWP in $H^{-1/2}(\mathbb{T})$ by Kenig Ponce and Vega [11]. Colliander, Keel, Staffilani, Takaoka and Tao [6] proved the global well-posedness in $H^{-1/2}(\mathbb{T})$ via the I-method. It was shown in [4] that the solution map cannot be smooth in $H^s(\mathbb{T})$ for $s < -1/2$. See [5] and [12] for the related results.

The idea of Bourgain's result [3] is using the conserved quantity $E_0$ to cancel the resonant parts with a loss of derivatives and applying the Fourier restriction norm method to the non-resonant parts to recover one derivatives. The Fourier restriction norm method is very useful to the study of the well-posedness for nonlinear dispersive wave equations. This method enables us to recover some derivatives in the non-resonant parts by using the weighted space-time Sobolev space $X^{s,b}$ whose norm (for the evolution operator $e^{-t\partial_x^3}$) is given by

$$\|u\|_{X^{s,b}} := \|e^{-t\partial_x^3} u\|_{H_t^b H_x^s} = \big\|\langle k \rangle^s \langle \tau - k^3 \rangle^b \widehat{u}(\tau, k)\big\|_{l_k^2 L_\tau^2}$$

where $\langle a \rangle = (1 + |a|)$ for $a \in \mathbb{R}$. Note that at most $j$ derivatives are recovered by the Fourier restriction norm method for the evolution operator $e^{-t\partial_x^{2j+1}}$. This fact implies that nonlinear terms of (1.1) have more derivatives than derivatives that can be recovered by the Fourier restriction norm method unlike to the KdV equation. That makes the problem difficult.

Babin, Ilyin and Titi [1] proved the well-posedness and unconditional uniqueness of (1.3) in $C([-T, T] : H^s)$ for $s \geq 0$. They applied the normal form reduction instead of the Fourier restriction norm method to recover one derivative. (We describe the idea of the normal form reduction in the latter part of this section.) We remark that at most one derivative can be recovered by the normal form reduction. Due to the present of three derivative losses in the nonlinearity of (1.1), this method does not work directly.

Recently, Kwak [13] proved the well-posedness of (1.1) with $\beta = 2\alpha$ in the energy space $H^2(\mathbb{T})$. To overcome difficulty mentioned above, he used the modified energy method introduced by Kwon [14]. Tsugawa [17] also applied the modified energy method to show the local well-posedness of higher order dispersive equations with smooth data.

This result includes in the local well-posedness of (1.1) without any condition on $\alpha$, $\beta$ and $\gamma$ for sufficiently large $s$ (see Proposition 6.4 below). See [9] for the related result.

In the present paper, we especially interested in the local well-posedness in low regularity and unconditional uniqueness of (1.1)–(1.2) because the analysis of low regularity could find the specific feature of the target equation, which might be hidden behind high regularity.

The proof of the main theorem is based on the following observation. As in [3], we use the conserved quantities $E_0$ and $E_1$ to absorb the resonant parts with derivative losses into the linear terms. Thus (1.1) with $\beta = 2\alpha$ can be rewritten into the following equation:

$$\partial_t u + \partial_x^5 u + \beta E_0(\varphi)\partial_x^3 u + 30\gamma E_1(\varphi)\partial_x u = J_1(u) + J_2(u) \tag{1.4}$$

where

$$J_1(u) = -30\gamma\Big(u^2 - \frac{1}{2\pi}\int_{\mathbb{T}} u^2\,dx\Big)\partial_x u,$$
$$J_2(u) = -\frac{\beta}{2}\partial_x(\partial_x u)^2 - \beta\partial_x\Big\{\Big(u - \frac{1}{2\pi}\int_{\mathbb{T}} u\,dx\Big)\partial_x^2 u\Big\}.$$

We need to apply the normal form reduction three times to (1.4) to recover derivative losses since at most one derivative can be recovered by using this method once. However, there are the cubic terms with two derivative losses appearing from $J_2(u)$ by the normal form reduction. We need to cancel those resonant parts with derivative losses to repeat the normal form reduction. Note that we can give the explicit formula of the resonant parts as follows:

$$\frac{\beta^2}{5}\Big\{\Big(\frac{1}{2\pi}\int_{\mathbb{T}} u^2\,dx\Big) - \Big(\frac{1}{2\pi}\int_{\mathbb{T}} u\,dx\Big)^2\Big\}.$$

(1.4) can be rewritten into the following equation because this resonant parts can be absorbed into the linear parts thanks to the conserved quantities $E_0$ and $E_1$:

$$\partial_t u + \partial_x^5 u + \beta E_0(\varphi)\partial_x^3 u + K(\varphi)\partial_x u = J_1(u) + J_2(u) + J_3(u) \tag{1.5}$$

where

$$J_3(u) = -\frac{\beta^2}{5}\Big\{\Big(\frac{1}{2\pi}\int_{\mathbb{T}} u^2\ dx\Big) - \Big(\frac{1}{2\pi}\int_{\mathbb{T}} u\ dx\Big)^2\Big\}\partial_x u,$$
$$K(\varphi) = \Big(30\gamma - \frac{\beta^2}{5}\Big)E_1(\varphi) + \frac{\beta^2}{5}E_0^2(\varphi).$$

$J_2(u)$ and $J_3(u)$ have no resonant parts with derivative losses. This implies that (1.1) has the symmetric structure. In this way, discovering the symmetric structure and cancellation properties are new ingredients in this paper.

When $u \in C([-T,T] : H^{3/2}(\mathbb{T}))$ and $\beta = 2\alpha$, the conservation laws $E_0(u(t)) = E_0(\varphi)$ and $E_1(u(t)) = E_1(\varphi)$ hold in the rigorous sense. We only need to show the local well-posedness of (1.5)–(1.2) because (1.1) with $\beta = 2\alpha$ is equivalent to (1.5). We consider the integral form of (1.5)–(1.2):

$$u(t) = U_\varphi(t)\varphi + \int_0^t U_\varphi(t-t') \sum_{j=1}^{3} J_j(u(t'))\, dt' \tag{1.6}$$

where $U_\varphi(t) := \mathcal{F}_k^{-1} \exp(t\phi_\varphi(k))\mathcal{F}_x$ and $\phi_\varphi(k) := -ik^5 + i\beta E_0(\varphi)k^3 - iK(\varphi)k$. We apply the normal form reduction three times to obtain the nonlinear terms with no losses. We use only $C([-T,T] : H^s(\mathbb{T}))$-norm to control all nonlinear terms obtained above when $s \geq 3/2$. As a result, from the contraction argument, the local well-posedness and unconditional uniqueness in $C([-T,T] : H^{3/2}(\mathbb{T}))$ can be obtained at the same time.

*Remark* 1.2. Due to the present of $\partial_x(\partial_x u)^2$ in (1.1), solutions cannot be defined in the distribution sense. This implies that the threshold is $s = 1$ in the sense of unconditional well-posedness. We expect to combine a gauge-like transformation and cancellation properties found in the present paper to obtain unconditional LWP in $H^1(\mathbb{T})$ (in the forthcoming paper by Tsugawa and the author).

In the following, we give the rigorous definition of the resonant parts and the non-resonant parts and describe the idea of the normal form reduction briefly. Before doing so, we introduce some notations. We write $k_{i,i+1,\cdots,j}$ to mean $k_i + k_{i+1} + \cdots + k_j$ for integers $i$ and $j$ such that $i < j$. For $N \in \mathbb{N}$ and $k \in \mathbb{Z}$, $\Gamma_k^{(N)}$ are denoted by

$$\Gamma_k^{(N)} = \{(k_1, k_2, \cdots, k_N) \in \mathbb{Z}^N \ ; \ k = k_1 + k_2 + \cdots + k_N\}.$$

Now we consider the integral equation as follows:

$$u(t) = U_\varphi(t)\varphi + \int_0^t U_\varphi(t-t')P^{(N)}(u, \partial_x u, \partial_x^2 u, \partial_x^3 u)(t')\, dt'$$

where $P^{(N)}$ is homogeneous polynomials of degree $N$. We apply the change of coordinate: $v(t) = U_\varphi(-t)u(t)$. Then $\hat{v}$ satisfies the following equation:

$$\widehat{v}(t,k) = \widehat{\varphi}(k) + \int_0^t \sum_{\Gamma_k^{(N)}} e^{-t'\Phi_\varphi^{(N)}} m^{(N)}(k_1, \cdots, k_N) \prod_{i=1}^{N} \widehat{v}(t', k_i)\, dt' \tag{1.7}$$

where $m^{(N)}$ is the $N$-multiplier corresponding to $P^{(N)}$. The phase function $\Phi_\varphi^{(N)}$ is defined by

$$\Phi_\varphi^{(N)} = \phi_\varphi(k_{1,2,\cdots,N}) - \sum_{j=1}^{N} \phi_\varphi(k_j).$$

The nonlinear terms corresponding the parts satisfied $\Phi_\varphi^{(N)} = 0$ are called resonant parts and the nonlinearities corresponding to the parts satisfied $\Phi_\varphi^{(N)} \neq 0$ are called the non-resonant parts.

When $\Phi_\varphi^{(N)} \neq 0$, we can integrate the Duhamel term of (1.7) by parts. If the multiplier $m^{(N)}$ is symmetric, we have

$$\begin{aligned}\int_0^t \sum_{\Phi_\varphi^{(N)}\neq 0} e^{-t'\Phi_\varphi^{(N)}} m^{(N)} \prod_{i=1}^N \widehat{v}(t',k_i)\,dt' &= -\Big[ \sum_{\Phi_\varphi^{(N)}\neq 0} e^{-t'\Phi_\varphi^{(N)}} \frac{m^{(N)}}{\Phi_\varphi^{(N)}} \prod_{i=1}^N \widehat{v}(t',k_i)\Big]_0^t \\ &\quad + N\int_0^t \sum_{\Phi_\varphi^{(N)}\neq 0} e^{-t'\Phi_\varphi^{(N)}} \frac{m^{(N)}}{\Phi_\varphi^{(N)}} \prod_{i=1}^{N-1} \widehat{v}(t',k_i)\partial_{t'}\widehat{v}(t',k_N)\,dt'.\end{aligned} \tag{1.8}$$

We substitute the differential form of (1.7) into $\partial_t \widehat{v}(t,k_N)$. This procedure is called the normal form reduction (see [16]).

In (1.8), both terms have $\Phi_\varphi^{(N)} (\neq 0)$ in the denominators, and this provides smoothing. Indeed, when only one variable ($|k_N|$) is the largest, $|\Phi_\varphi^{(N)}|$ is grater than $|k_N|^4$ (see Lemmas 2.1–2.3 below). This property enables us to recover one derivative by applying the normal form reduction to (1.6) because $\partial_t v$ has three derivatives in (1.6). Since the resonant parts with derivative losses are removed from (1.6) as mentioned above, we can apply the normal form method three times to (1.6) to eliminate derivative losses.

Finally, we give some notations. We will use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C$ and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. For a Banach space $X$, we define $B_r(X) = \{f \in X \mid \|f\|_X \leq r\}$. The rest of this paper is planned as follows. In Section 2, we introduce some notations and preliminary lemmas. In Section 3, we rewrite (1.6) by the normal form reduction. In Section 4, we give a variant of the Sobolev embedding theorems. In Section 5, we establish an appropriate upper bounds of the multipliers corresponding to nonlinear terms appearing from (1.6) by the normal form reduction. In section 6, we give the proof of the main theorem.

## § 2. notations and preliminary lemmas

In this section, we prepare some lemmas and propositions to prove the main theorem. First, we give some notations. For a $(2\pi)$-periodic function $f$ and a function $g$ on $\mathbb{Z}$, we define the Fourier transform and the inverse Fourier transform by

$$(\mathcal{F}_x f)(k) := \widehat{f}(k) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ixk} f(x)\,dx, \quad (\mathcal{F}_k^{-1} g)(x) := \sum_{k\in\mathbb{Z}} e^{ixk} g(k).$$

Then, we have

$$f(x) = \mathcal{F}_k^{-1}(\mathcal{F}_x f), \quad \|f\|_{L^2} := \Big(\frac{1}{2\pi}\int_{\mathbb{T}} |f|^2\,dx\Big)^{1/2} = \Big(\sum_{k\in\mathbb{Z}} |\widehat{f}|^2\,dk\Big)^{1/2}.$$

We give some estimate on the phase function $\Phi_\varphi^{(N)}$ defined as

$$\Phi_\varphi^{(N)} = \phi_\varphi(k_{1,2,\cdots,N}) - \sum_{j=1}^{N} \phi_\varphi(k_j) \quad \text{for } N \in \mathbb{N},$$

which plays an important role to recover some derivatives when we estimate non-resonant parts of nonlinear terms. A simple calculation yields that $\Phi_f^{(1)} = 0$,

$$\Phi_f^{(2)} = -\frac{5}{2} i k_1 k_2 k_{1,2} \Big( k_1^2 + k_2^2 + k_{1,2}^2 - \frac{6}{5}\beta E_0(f) \Big), \tag{2.1}$$

$$\Phi_f^{(3)} = -\frac{5}{2} i k_{1,2} k_{2,3} k_{3,1} \Big( k_{1,2}^2 + k_{2,3}^2 + k_{3,1}^2 - \frac{6}{5}\beta E_0(f) \Big). \tag{2.2}$$

From these identities, we obtain the following lemmas.

**Lemma 2.1.** *Assume that $f, g \in L^2(\mathbb{T})$, $|k_2| \gtrsim |k_1|$, $|k_2| \gtrsim |\beta E_0(f)|$ and $k_1 k_2 k_{1,2} \neq 0$. Then, we have*

$$\begin{gathered} |\Phi_f^{(2)}| \gtrsim \min\{|k_1|, |k_{1,2}|\} |k_2|^4, \\ \Big| \frac{1}{\Phi_f^{(2)}} - \frac{1}{\Phi_g^{(2)}} \Big| \lesssim \frac{|\beta| |E_0(f) - E_0(g)|}{|k_2|^2 |\Phi_g^{(2)}|} \end{gathered} \tag{2.3}$$

**Lemma 2.2.** *Suppose that $f, g \in L^2(\mathbb{T})$. If $|k_3| \gg |k_2| \gtrsim |k_1|$, $|k_3| \gtrsim |\beta E_0(f)|$ and $k_{1,2} \neq 0$, then we have*

$$\begin{gathered} |\Phi_f^{(3)}| \gtrsim |k_{1,2}| |k_3|^4, \\ \Big| \frac{1}{\Phi_f^{(3)}} - \frac{1}{\Phi_g^{(3)}} \Big| \lesssim \frac{|\beta| |E_0(f) - E_0(g)|}{|k_3|^2 |\Phi_g^{(3)}|}. \end{gathered} \tag{2.4}$$

*If $|k_3| \sim |k_2| \gg |k_1|$, $|k_3| \gtrsim |\beta E_0(f)|$ and $k_{2,3} \neq 0$, then we have*

$$\begin{gathered} |\Phi_f^{(3)}| \gtrsim |k_{2,3}| |k_3|^4, \\ \Big| \frac{1}{\Phi_f^{(3)}} - \frac{1}{\Phi_g^{(3)}} \Big| \lesssim \frac{|\beta| |E_0(f) - E_0(g)|}{|k_3|^2 |\Phi_g^{(3)}|}. \end{gathered}$$

We need to estimates similar to (2.3) and (2.4) for $N \geq 4$ to recover derivative losses. However $\Phi_f^{(N)}$ cannot be factorized exactly when $N \geq 4$. In general, the following does not hold:

$$|\Phi_f^{(4)}| \gtrsim |k_{1,2,3}| |k_4|^4$$

when $|k_4| \gtrsim |\beta E_0(f)|$, $k_{1,2,3} \neq 0$ and

$$|k_4| \gg |k_3| \gtrsim |k_2| \gtrsim |k_1|. \tag{2.5}$$

In fact, we easily check that

$$\begin{aligned}|\Phi_f^{(4)}| &\lesssim |(n^{12}+1)^5-(n^{12})^5-(n^{11}+n^4)^5-(-n^{11})^5-(-n^4+1)^5| \\ &\sim n^{39} \ll |k_4|^4\end{aligned}$$

when $k_4 = n^{12}$, $k_3 = n^{11}+n^4$, $k_2 = -n^{11}$, $k_1 = -n^4+1$ and $n$ is sufficiently large. Thus, we need the slightly stronger assumption (2.6) or (2.7) instead of (2.5) in the following lemma.

**Lemma 2.3.** *Assume that $f, g \in L^2(\mathbb{T})$, $|k_N| \gtrsim |\beta E_0(f)|$, $k_{1,2,\cdots,N-1} \neq 0$ and*

$$|k_N| \gg |k_{N-1}| \gg |k_{N-2}| \gtrsim \cdots \gtrsim |k_1| \tag{2.6}$$

*or*

$$|k_N|^{4/5} \gg |k_{N-1}| \gtrsim |k_{N-2}| \gtrsim \cdots \gtrsim |k_1|. \tag{2.7}$$

*Then it follows that*

$$\begin{gathered}|\Phi_f^{(N)}| \gtrsim |k_{1,2,\cdots,N-1}||k_N|^4, \\ \Big|\frac{1}{\Phi_f^{(N)}}-\frac{1}{\Phi_g^{(N)}}\Big| \lesssim \frac{|\beta||E_0(f)-E_0(g)|}{|k_N||\Phi_g^{(N)}|}.\end{gathered} \tag{2.8}$$

*Proof.* A direct computation shows that

(2.9)

$$\begin{aligned}&\Big|k_{1,2,\cdots,N}^5-\sum_{j=1}^N k_j^5\Big| \\ &= \Big|5k_{1,2,\cdots,N-1}k_N^4+10k_{1,2,\cdots,N-1}^2k_N^3++10k_{1,2,\cdots,N-1}^3k_N^2+5k_{1,2,\cdots,N-1}^4k_N-\sum_{j=1}^{N-1}k_j^5\Big| \\ &\gtrsim |k_{1,2,\cdots,N-1}||k_N|^4-C|k_{N-1}|^5.\end{aligned}$$

Note that a simple calculation yields that

$$|k_{1,2,\cdots,N-1}||k_N|^4 \gtrsim |k_{N-1}||k_N|^4 \gg |k_{N-1}|^5 \tag{2.10}$$

when (2.6) holds. Moreover, it follows that

$$|k_{1,2,\cdots,N-1}||k_N|^4 \geq |k_N|^4 \gg |k_{N-1}|^5 \tag{2.11}$$

when (2.7) holds. Collecting with (2.9)–(2.11), we obtain

$$\begin{aligned}|\Phi_f^{(N)}| =&\Big|-i\big(k_{1,2,\cdots,N}^5-\sum_{j=1}^N k_j^5\big)+i\beta E_0(f)\big(k_{1,2,\cdots,N}^3-\sum_{j=1}^N k_j^3\big)\Big| \\ \gtrsim&|k_{1,2,\cdots,N-1}||k_N|^4-C|k_{N-1}|^5-C|\beta E_0(f)||k_N|^3 \\ \gtrsim&|k_{1,2,\cdots,N-1}||k_N|^4.\end{aligned}$$

Form the above inequality, we obtain

$$\Big|\frac{1}{\Phi_f^{(N)}} - \frac{1}{\Phi_g^{(N)}}\Big| \le \frac{|\beta||E_0(f)-E_0(g)|\big|k_{1,2,\cdots,N}^3 - \sum_{j=1}^N k_j^3\big|}{|\Phi_f^{(N)}||\Phi_g^{(N)}|}$$
$$\lesssim \frac{|\beta||E_0(f)-E_0(g)||k_N|^3}{|k_{1,\cdots,N-1}||k_N|^4|\Phi_g^{(N)}|} \le \frac{|\beta||E_0(f)-E_0(g)|}{|k_N||\Phi_g^{(N)}|}$$

□

**Lemma 2.4.** *Fix $t\in\mathbb{R}$ and $(k_1,\cdots,k_N)\in\Gamma_k^{(N)}$. For $f,g\in L^2(\mathbb{T})$, we have $|e^{-t\Phi_f^{(N)}}|=1$ and $|e^{-t\Phi_f^{(N)}} - e^{-t\Phi_g^{(N)}}|\to 0$ when $E_0(f)\to E_0(g)$.*

*Proof.* Since $\operatorname{Re}\Phi_f^{(N)}=0$, we have $|e^{-t\Phi_f^{(N)}}|=1$. Note that

$$\Phi_f^{(N)} - \Phi_g^{(N)} = i\beta(E_0(f)-E_0(g))\big(k_{1,2,\cdots,N}^3 - \sum_{j=1}^N k_j^3\big).$$

A simple calculation yields that

$$|e^{-t\Phi_f^{(N)}} - e^{-t\Phi_g^{(N)}}| = |e^{-t\Phi_f^{(N)}}||1-e^{-t(\Phi_g^{(N)}-\Phi_f^{(N)})}|$$
$$\le C|t||\beta||E_0(f)-E_0(g)|\big|k_{1,2,\cdots,N}^3 - \sum_{j=1}^N k_j^3\big| \to 0$$

when $E_0(f)\to E_0(g)$. □

*Definition* 1. For $f\in L^1(\mathbb{T})$ and an $N$-multiplier $m^{(N)}(k_1,\cdots,k_N)$, we define a $N$-linear functional $\Lambda_f^{(N)}$ by

$$\Lambda_f^{(N)}(m^{(N)},\widehat{v}_1,\cdots,\widehat{v}_N)(t,k)$$
$$= \sum_{(k_1,\cdots,k_N)\in\Gamma_k^{(N)}} e^{-t\Phi_f^{(N)}} m^{(N)}(k_1,\cdots,k_N)\widehat{v}_1(k_1)\cdots\widehat{v}_N(k_N)$$

where $\widehat{v}_1,\cdots,\widehat{v}_N$ are functions on $\mathbb{Z}^N$. $\Lambda_f^{(N)}(m^{(N)},\widehat{v},\widehat{v},\cdots,\widehat{v})$ may simply be written $\Lambda_f^{(N)}(m^{(N)},\widehat{v})$.

*Definition* 2. We say an $N$-multiplier $m^{(N)}$ is symmetric if

$$m^{(N)}(k_1,\cdots,k_N) = m^{(N)}(k_{\sigma(1)},\cdots,k_{\sigma(N)})$$

for all $\sigma\in S_N$, the group of all permutations on $N$ objects. The symmetrization of an $N$-multiplier $m^{(N)}$ is defined as

$$[m^{(N)}]_{sym}^{(N)}(k_1,\cdots,k_N) = \frac{1}{N!}\sum_{\sigma\in S_N} m^{(N)}(k_{\sigma(1)},\cdots,k_{\sigma(N)}).$$

We define the $(N+j)$-extension operators of a $N$-multiplier $m^{(N)}$ by

$$[m^{(N)}]_{ext1}^{(N+j)}(k_1,\cdots,k_{N+j}) = m^{(N)}(k_1,\cdots,k_{N-1},k_{N,\cdots,N+j}),$$
$$[m^{(N)}]_{ext2}^{(N+j)}(k_1,\cdots,k_{N+j}) = m^{(N)}(k_{1+j},\cdots,k_{N+j})$$

for $j \in \mathbb{N}$.

A simple calculation yields that it follows that

$$\begin{aligned}\Lambda^{(j)}(m^{(j)},\widehat{v},\cdots,\widehat{v},\Lambda^{(l)}(m_*^{(l)},\widehat{v})) &= \Lambda^{(j+l-1)}([m^{(j)}]_{ext1}^{(j+l-1)}[m_*^{(l)}]_{ext2}^{(j+l-1)},\widehat{v}) \\ &= \Lambda^{(j+l-1)}(\big[[m^{(j)}]_{ext1}^{(j+l-1)}[m_*^{(l)}]_{ext2}^{(j+l-1)}\big]_{sym}^{(j+l-1)},\widehat{v}).\end{aligned}$$

for any symmetric multipliers $m^{(j)}$, $m_*^{(l)}$.

Let $N,l \in \mathbb{N}$ such that $N \geq 2$ and $2 \leq l \leq N$ . For $L > 0$, we define multipliers to restrict summation regions in the Fourier space as follows:

$$m_{\leq L}^{(N)} := \begin{cases} 1, \text{when } \max_{1\leq j\leq N}\{|k_j|\} \leq L \\ 0, \text{otherwise} \end{cases}, \quad m_{>L}^{(N)} := \begin{cases} 1, \text{when } \max_{1\leq j\leq N}\{|k_j|\} > L \\ 0, \text{otherwise} \end{cases},$$

$$m_{Hl}^{(N)} := \begin{cases} 1, \text{when } |k_N| \sim |k_{N-1}| \sim \cdots \sim |k_{N-l+1}| \gg \max_{1\leq j\leq N-l}\{|k_j|\} \\ 0, \text{otherwise} \end{cases},$$

$$m_{H1}^{(N)} := \begin{cases} 1, \text{when } |k_N| \geq 2^N \max_{1\leq i\leq N-1}\{|k_j|\} \\ 0, \text{otherwise} \end{cases},$$

$$m_{H2}^{(2)} := \begin{cases} 1, \text{when } |k_1|/4 \leq |k_2| \leq 4|k_1| \\ 0, \text{otherwise} \end{cases},$$

$$m_{NR}^{(2)} := \begin{cases} 1, \text{when } k_1k_2k_{1,2} \neq 0 \\ 0, \text{otherwise} \end{cases}, \quad m_{NR}^{(3)} := \begin{cases} 1, \text{when } k_{1,2}k_{2,3}k_{3,1} \neq 0 \\ 0, \text{otherwise} \end{cases},$$

$$m_{R1}^{(N)} := \begin{cases} 1, \text{when } k_{1,2,\cdots,N-1} = 0 \\ 0, \text{otherwise} \end{cases}, \quad m_{R2}^{(N)} := \begin{cases} 1, \text{when } k_1k_2\cdots k_N = 0 \\ 0, \text{otherwise} \end{cases},$$

$$m_{R3}^{(3)} := \begin{cases} 1, \text{when } k_1 = -k_2 = k_3 \\ 0, \text{otherwise} \end{cases}, \quad m_{R4}^{(N)} := \begin{cases} 1, \text{when } k_1 = k_2 = \cdots = k_N = 0 \\ 0, \text{otherwise} \end{cases},$$

$$m_{R5}^{(4)} := \begin{cases} 1, \text{when } |k_3|^{5/4} \gtrsim |k_4| \gg |k_3| \sim |k_2| \gtrsim |k_1|, \quad k_{1,2,3} \neq 0 \\ 0, \text{otherwise} \end{cases},$$

Put

$$q_1^{(2)} = -ik_{1,2}(k_1^2 + k_2^2 + k_{1,2}^2), \quad q_2^{(3)} = ik_{1,2,3}, \quad Q_1^{(2)} = \frac{\beta}{4} q_2^{(2)},$$
$$Q_2^{(3)} = q_2^{(3)}\{10\gamma m_{NR}^{(3)} + \frac{\beta^2}{5}[m_{R1}^{(3)}(1 - m_{R2}^{(3)})]_{sym}^{(3)} - 10\gamma[3m_{R3}^{(3)}]_{sym}^{(3)}\}.$$

Then, we have the following proposition.

**Proposition 2.5.** *Let $N \in \mathbb{N}, s \geq 0$, $m^{(N)}$ be a symmetric and $u \in C([-T,T] : H^s(\mathbb{T}))$ satisfy (1.6). Then, it follows that $v := \mathcal{F}_k^{-1}(e^{-t\phi_\varphi(k)}\widehat{u}(t,k)) \in C([-T,T] : H^s(\mathbb{T}))$ and $\widehat{v}(t,k)$ satisfies*

$$\begin{aligned}\Big[\Lambda_\varphi^{(N)}(m^{(N)}, \widehat{v})(t')\Big]_0^t = &\int_0^t \Lambda_\varphi^{(N)}(-\Phi_\varphi^{(N)} m^{(N)}, \widehat{v})(t') \\ &+\Lambda_\varphi^{(N+1)}\big(\big[[Nm^{(N)}]_{ext1}^{(N+1)}[-Q_1^{(2)} m_{NR}^{(2)}]_{ext2}^{(N+1)}\big]_{sym}^{(N+1)}, \widehat{v}(t')\big) \\ &+\Lambda_\varphi^{(N+2)}\big(\big[[Nm^{(N)}]_{ext1}^{(N+2)}[-Q_2^{(3)}]_{ext2}^{(N+2)}\big]_{sym}^{(N+2)}, \widehat{v}(t')\big)\, dt'\end{aligned} \tag{2.12}$$

*on $[-T,T]$.*

*Proof.* First, we will prove (2.12) with $N = 1, m^{(1)} = 1$:

$$\Big[\widehat{v}(t')\Big]_0^t = \int_0^t \Lambda_\varphi^{(2)}\big(-Q_1^{(2)} m_{NR}^{(2)}, \widehat{v}(t')\big) + \Lambda_\varphi^{(3)}\big(-Q_2^{(3)}, \widehat{v}(t')\big)dt'. \tag{2.13}$$

From the definition of $U_\varphi$, we have

$$e^{-t\phi_\varphi(k)}\mathcal{F}_x\big[u(t) - U_\varphi(t)\varphi\big] = [\hat{v}(t')]_0^t, \tag{2.14}$$
$$e^{-t\phi_\varphi(k)}\mathcal{F}_x\big[U_\varphi(t-t')J_j(u(t'))\big] = e^{-t'\phi_\varphi(k)}\mathcal{F}_x\big[J_j(u(t'))\big]. \tag{2.15}$$

Clearly, we have

$$\begin{aligned}&e^{-t'\phi_\varphi(k)}\mathcal{F}_x\big[-\frac{\beta}{2}\partial_x((\partial_x u)^2) - \beta\partial_x(u\partial_x^2 u)\big] \\ &\quad= \sum_{\Gamma_k^{(2)}} e^{-t'\Phi_\varphi^{(2)}}\big(-\frac{\beta}{4}q_1^{(2)}\big)\prod_{j=1}^2 \hat{v}(t',k_j) = \sum_{\Gamma_k^{(2)}} e^{-t'\Phi_\varphi^{(2)}}(-Q_1^{(2)})\prod_{j=1}^2 \hat{v}(t',k_j)\end{aligned}$$

and

$$e^{-t'\phi_\varphi(k)}\mathcal{F}_x\Big[\frac{\beta}{2\pi}\int_{\mathbb{T}} u\; dx\partial_x^3 u\Big] = \sum_{\Gamma_k^{(2)}} e^{-t'\Phi_\varphi^{(2)}} Q_1^{(2)}[2m_{R1}^{(2)}]_{sym}^{(2)}\prod_{j=1}^2 \hat{v}(t',k_j)$$

Note that $m_{NR}^{(2)} = 1 - [2m_{R1}^{(2)}]_{sym}^{(2)} + m_{R4}^{(2)}$ and $Q_1^{(2)} m_{R4}^{(2)} = 0$. Thus we obtain

$$\begin{aligned} e^{-t'\phi_\varphi(k)} \mathcal{F}_x[J_2(u(t'))] &= \sum_{\Gamma_k^{(2)}} e^{-t'\Phi_\varphi^{(2)}} (-Q_1^{(2)})(1 - [2m_{R1}^{(2)}]_{sym}^{(2)}) \prod_{j=1}^{2} \hat{v}(t', k_j) \\ &= \sum_{\Gamma_k^{(2)}} e^{-t'\Phi_\varphi^{(2)}} (-Q_1^{(2)}) m_{NR}^{(2)} \prod_{j=1}^{2} \hat{v}(t', k_j) \\ &= \Lambda_\varphi^{(2)}(-Q_1^{(2)} m_{NR}^{(2)}, \hat{v}(t')). \end{aligned} \tag{2.16}$$

Since

$$e^{-t'\phi_\varphi(k)} \mathcal{F}_x\Big[\frac{1}{2\pi}\int_{\mathbb{T}} u^2\, dx\, \partial_x u\Big] = \sum_{\Gamma_k^{(3)}} e^{-t'\Phi_\varphi^{(3)}} q_2^{(3)} [m_{R1}^{(3)}]_{sym}^{(3)} \prod_{j=1}^{3} \hat{v}(t', k_j)$$

and

$$e^{-t'\phi_\varphi(k)} \mathcal{F}_x\Big[\Big(\frac{1}{2\pi}\int_{\mathbb{T}} u\, dx\Big)^2 \partial_x u\Big] = \sum_{\Gamma_k^{(3)}} e^{-t'\Phi_\varphi^{(3)}} q_2^{(3)} [m_{R1}^{(3)} m_{R2}^{(3)}]_{sym}^{(3)} \prod_{j=1}^{3} \hat{v}(t', k_j),$$

we obtain

$$\begin{aligned} e^{-t'\phi_\varphi(k)} \mathcal{F}_x[J_3(u(t'))] &= \sum_{\Gamma_k^{(3)}} e^{-t'\Phi_\varphi^{(3)}} \big(-\frac{\beta^2}{5}\big) q_2^{(3)} [m_{R1}^{(3)}(1 - m_{R2}^{(3)})]_{sym}^{(3)} \prod_{j=1}^{3} \hat{v}(t', k_j) \\ &= \Lambda_\varphi^{(3)}\big(-\frac{\beta^2}{5} q_2^{(3)} [m_{R1}^{(3)}(1 - m_{R2}^{(3)})]_{sym}^{(3)}, \hat{v}(t')\big) \end{aligned} \tag{2.17}$$

and

(2.18)
$$\begin{aligned} e^{-t'\phi_\varphi(k)} \mathcal{F}_x[J_1(u(t'))] &= \sum_{\Gamma_k^{(3)}} e^{-t'\Phi_\varphi^{(3)}} (-10\gamma) q_2^{(3)} (1 - [3m_{R1}^{(3)}]_{sym}^{(3)}) \prod_{j=1}^{3} \hat{v}(t', k_j) \\ &= \sum_{\Gamma_k^{(3)}} e^{-t'\Phi_\varphi^{(3)}} (-10\gamma) q_2^{(3)} (m_{NR}^{(3)} - [3m_{R3}^{(3)}]_{sym}^{(3)}) \prod_{j=1}^{3} \hat{v}(t', k_j) \\ &= \Lambda_\varphi^{(3)}(-10\gamma q_2^{(3)} m_{NR}^{(3)}, \hat{v}(t')) + \Lambda_\varphi^{(3)}(10\gamma q_2^{(3)} [3m_{R3}^{(3)}]_{sym}^{(3)}, \hat{v}(t')) \end{aligned}$$

In the second equality, we used $m_{NR}^{(3)} = 1 - [3m_{R1}^{(3)}]_{sym}^{(3)} + [3m_{R3}^{(3)}]_{sym}^{(3)} + m_{R4}^{(3)}$ and $q_2^{(3)} m_{R4}^{(3)} = 0$.

Therefore, collecting (2.14)–(2.18), we have (2.13). We differentiate (2.13) to obtain

$$\frac{d}{dt}\hat{v}(t,k) = \sum_{\Gamma_k^{(2)}} e^{-t\Phi_\varphi^{(2)}} (-Q_1^{(2)} m_{NR}^{(2)}) \prod_{i=1}^{2} \hat{v}(t, k_i) + \sum_{\Gamma_k^{(3)}} e^{-t\Phi_\varphi^{(3)}} (-Q_2^{(3)}) \prod_{i=1}^{3} \hat{v}(t, k_i). \tag{2.19}$$

On the other hand, we use the Leibniz rule to have

$$
\begin{aligned}
\big[\Lambda_{\varphi}^{(N)}(m^{(N)},\hat{v}(t'))\big]_0^t &= \int_0^t \Lambda_{\varphi}^{(N)}\big(-\Phi_{\varphi}^{(N)}m^{(N)},\hat{v}(t')\big)\,dt' \\
&\quad + \int_0^t \sum_{\Gamma_k^{(N)}} e^{-t\Phi_{\varphi}^{(N)}}\, Nm^{(N)} \prod_{i=1}^{N-1} \hat{v}(t',k_i)\frac{d}{dt'}\hat{v}(t',k_N)\,dt'
\end{aligned} \tag{2.20}
$$

Substituting (2.19) into (2.20), we obtain (2.12). □

## § 3. The normal form reductions

In this section, we will remove the nonlinear terms with derivative losses from (1.6) by the normal form reduction.

First, We rewrite (1.6) by the normal form reduction.

**Proposition 3.1.** *Let $s \geq 0$ and $u \in C([-T,T]:H^s(\mathbb{T}))$ be a solution of (1.6). Then, it follows that $v(t) = \mathcal{F}_k^{-1}\big[e^{-t\phi_\varphi(k)}\hat{u}(t,k)\big] \in C([-T,T]:H^s(\mathbb{T}))$ and*

$$
\big[v(t') + F_{\varphi,L}(v(t'))\big]_0^t = \int_0^t G_{\varphi,L}(v(t'))\,dt' \tag{3.1}
$$

*where*

$$
\begin{aligned}
\widehat{F}_{\varphi,L}(\widehat{v}) &= \sum_{j=1}^{2}\Lambda_{\varphi}^{(2)}(\tilde{L}_{j,\varphi}^{(2)}m_{>L}^{(2)},\widehat{v}) + \sum_{j=1}^{3}\Lambda_{\varphi}^{(3)}(\tilde{L}_{j,\varphi}^{(3)}m_{>L}^{(3)},\widehat{v}) + \Lambda_{\varphi}^{(4)}(\tilde{L}_{1,\varphi}^{(4)}m_{>L}^{(4)},\widehat{v}),\\
\widehat{G}_{\varphi,L}(\widehat{v}) &= \sum_{j=1}^{2}\Lambda_{\varphi}^{(2)}(\tilde{L}_{j,\varphi}^{(2)}\Phi_{\varphi}^{(2)}m_{\leq L}^{(2)},\widehat{v}) + \sum_{j=1}^{3}\Lambda_{\varphi}^{(3)}(\tilde{L}_{j,\varphi}^{(3)}\Phi_{\varphi}^{(3)}m_{\leq L}^{(3)},\widehat{v}) + \Lambda_{\varphi}^{(4)}(\tilde{L}_{1,\varphi}^{(4)}\Phi_{\varphi}^{(4)}m_{\leq L}^{(4)},\widehat{v})\\
&\quad + \sum_{j=1}^{10}\Lambda_{\varphi}^{(3)}(\tilde{M}_{j,\varphi}^{(3)},\widehat{v}) + \sum_{j=1}^{8}\Lambda_{\varphi}^{(4)}(\tilde{M}_{j,\varphi}^{(4)},\hat{v}) + \sum_{j=1}^{2}\Lambda_{\varphi}^{(5)}(\tilde{M}_{j,\varphi}^{(5)},\widehat{v}) + \Lambda_{\varphi}^{(6)}(\tilde{M}_{1,\varphi}^{(6)},\widehat{v}).
\end{aligned}
$$

*Here, for $j,N \in \mathbb{N}$, the multipliers $\tilde{L}_{j,\varphi}^{(N)}$ and $\tilde{M}_{j,\varphi}^{(N)}$ are symmetrization of $L_{j,\varphi}^{(N)}$ and $M_{j,\varphi}^{(N)}$ respectively. These multipliers are defined as follows:*

$$
\begin{aligned}
&L_{1,\varphi}^{(2)} = -Q_1^{(2)}m_{NR}^{(2)}\,2m_{H1}^{(2)}/\Phi_{\varphi}^{(2)}, \quad L_{2,\varphi}^{(2)} = -Q_1^{(2)}m_{NR}^{(3)}m_{H2}^{(2)}/\Phi_{\varphi}^{(2)},\\
&L_{1,\varphi}^{(3)} = -10\gamma q_2^{(3)}m_{NR}^{(3)}3m_{H1}/\Phi_{\varphi}^{(3)},\\
&L_{2,\varphi}^{(3)} = [2\tilde{L}_{1,0}^{(2)}]_{ext1}^{(3)}[-Q_1^{(2)}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)}]_{ext2}^{(3)}2m_{H1}^{(3)}(1-m_{R1}^{(3)})/\Phi_{\varphi}^{(3)},\\
&L_{3,\varphi}^{(3)} = [2\tilde{L}_{1,0}^{(2)}m_{>L}^{(2)}]_{ext1}^{(3)}[-Q_1^{(2)}m_{NR}^{(2)}m_{H2}^{(2)}]_{ext2}^{(3)}m_{H2}^{(3)}/\Phi_{\varphi}^{(3)},\\
&L_{1,\varphi}^{(4)} = \big[3\tilde{L}_{2,0}^{(3)}\big]_{ext1}^{(4)}\big[-Q_1^{(2)}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(3)}2m_{H1}^{(4)}(1-m_{R1}^{(4)})(1-m_{R5}^{(4)})/\Phi_{\varphi}^{(4)}
\end{aligned}
$$

*and*

$$
\begin{aligned}
M_1^{(3)} &= 10\gamma q_2^{(3)} 3m_{R3}^{(3)}, \quad M_2^{(3)} = -\frac{\beta^2}{5} q_2^{(3)} m_{H1}^{(3)} m_{R1}^{(3)} (1 - m_{R2}^{(3)}),\\
M_{3,\varphi}^{(3)} &= -q_2^{(3)} \big(10\gamma m_{NR}^{(3)} (1 - [3m_{H1}^{(3)}]_{sym}^{(3)}) + \frac{\beta^2}{5} m_{R1}^{(3)} (1 - m_{H1}^{(3)})(1 - m_{R2}^{(3)})\big),\\
M_{4,\varphi}^{(3)} &= \big[2\tilde{L}_{1,\varphi}^{(2)} m_{>L}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(3)} (1 - 2m_{H1}^{(3)}),\\
M_{5,\varphi}^{(3)} &= \big[2(\tilde{L}_{1,\varphi}^{(2)} - \tilde{L}_{1,0}^{(2)}) m_{>L}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(3)} 2m_{H1}^{(3)},\\
M_{6,\varphi}^{(3)} &= \big[-2\tilde{L}_{1,0}^{(2)} m_{\le L}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(3)} 2m_{H1}^{(3)},\\
M_{7,\varphi}^{(3)} &= \big[2\tilde{L}_{1,0}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(3)} 2m_{H1}^{(3)} m_{R1}^{(3)},\\
M_{8,\varphi}^{(3)} &= \big[2\tilde{L}_{1,0}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)} m_{H2}^{(2)}\big]_{ext2}^{(3)} (1 - m_{H2}^{(3)}),\\
M_{9,\varphi}^{(3)} &= \big[2(\tilde{L}_{1,\varphi}^{(2)} - \tilde{L}_{1,0}^{(2)}) m_{>L}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)} m_{H2}^{(3)}\big]_{ext2}^{(3)} m_{H2}^{(3)},\\
M_{10,\varphi}^{(3)} &= \big[2\tilde{L}_{2,\varphi}^{(2)} m_{>L}^{(2)}\big]_{ext1}^{(3)} \big[-Q_1^{(2)} m_{NR}^{(2)}\big]_{ext2}^{(3)},
\end{aligned}
$$

$$
\begin{aligned}
M_{1,\varphi}^{(4)} &= \big[2(\tilde{L}_{1,\varphi}^{(2)} + \tilde{L}_{2,\varphi}^{(2)}) m_{>L}^{(2)}\big]_{ext1}^{(4)} \big[-Q_2^{(3)}\big]_{ext2}^{(4)},\\
M_{2,\varphi}^{(4)} &= [3(\tilde{L}_{1,\varphi}^{(3)} + \tilde{L}_{3,\varphi}^{(3)}) m_{>L}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)}]_{ext2}^{(4)},\\
M_{3,\varphi}^{(4)} &= [3\tilde{L}_{2,\varphi}^{(3)} m_{>L}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}]_{ext2}^{(4)} (1 - 2m_{H1}^{(4)}),\\
M_{4,\varphi}^{(4)} &= [3(\tilde{L}_{2,\varphi}^{(3)} - \tilde{L}_{2,0}^{(3)}) m_{>L}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}]_{ext2}^{(4)} 2m_{H1}^{(4)},\\
M_{5,\varphi}^{(4)} &= [-3\tilde{L}_{2,0}^{(3)} m_{\le L}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}]_{ext2}^{(4)} 2m_{H1}^{(4)},\\
M_{6,\varphi}^{(4)} &= [3\tilde{L}_{2,0}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}]_{ext2}^{(4)} 2m_{H1}^{4} m_{R1}^{(4)},\\
M_{7,\varphi}^{(4)} &= [3\tilde{L}_{2,0}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}]_{ext2}^{(4)} 2m_{H1}^{(4)} m_{R5}^{(4)},\\
M_{8,\varphi}^{(4)} &= [3\tilde{L}_{2,\varphi}^{(3)} m_{>L}^{(3)}]_{ext1}^{(4)} [-Q_1^{(2)} m_{NR}^{(2)} m_{H2}^{(2)}]_{ext2}^{(4)},\\
M_{1,\varphi}^{(5)} &= \big[3\sum_{j=1}^{3} \tilde{L}_{j,\varphi}^{(3)} m_{>L}^{(3)}\big]_{ext1}^{(5)} [-Q_2^{(3)}]_{ext2}^{(5)},\\
M_{2,\varphi}^{(5)} &= \big[4\tilde{L}_{1,\varphi}^{(4)} m_{>L}^{(4)}\big]_{ext1}^{(5)}, \big[-Q_1^{(2)} m_{NR}^{(2)}\big]_{ext2}^{(5)},\\
M_{1,\varphi}^{(6)} &= \big[4\tilde{L}_{1,\varphi}^{(4)} m_{>L}^{(4)}\big]_{ext1}^{(6)} \big[-Q_1^{(3)}\big]_{ext2}^{(6)}.
\end{aligned}
$$

*Proof.* Obviously, we have

$$\tilde{L}_{1,\varphi}^{(2)} \Phi_\varphi^{(2)} m_{>L}^{(2)} + \tilde{L}_{1,\varphi}^{(2)} \Phi_\varphi^{(2)} m_{\le L}^{(2)} = -Q_1^{(2)} m_{NR}^{(2)} [2m_{H1}^{(2)}]_{sym}^{(2)}, \tag{3.2}$$

$$\tilde{L}_{2,\varphi}^{(2)} \Phi_\varphi^{(2)} m_{>L}^{(2)} + \tilde{L}_{2,\varphi}^{(2)} \Phi_\varphi^{(2)} m_{\le L}^{(2)} = -Q_1^{(2)} m_{NR}^{(2)} m_{H2}^{(3)} \tag{3.3}$$

and

$$\tilde{M}_{1,\varphi}^{(3)} + \tilde{M}_{2,\varphi}^{(3)} + \tilde{M}_{3,\varphi}^{(3)} + \tilde{L}_{1,\varphi}^{(3)} \Phi_\varphi^{(3)} m_{>L}^{(3)} + \tilde{L}_{1,\varphi}^{(3)} \Phi_\varphi^{(3)} m_{\le L} = -Q_2^{(3)}. \tag{3.4}$$

Since

$$[2m_{H1}^{(2)}]_{sym}^{(2)} + m_{H2}^{(2)} = 1, \tag{3.5}$$

and Proposition 2.5 with $m^{(2)} = \tilde{L}_{1,\varphi}^{(2)} m_{>L}^{(2)}$ and $N = 2$, it follows that

$$\begin{aligned}
&\int_0^t \Lambda_\varphi^{(2)}\big(\tilde{L}_{1,\varphi}^{(2)}\Phi_\varphi^{(2)} m_{>L}^{(2)}, \widehat{v}(t')\big)\,dt' + \big[\Lambda_\varphi^{(2)}\big(\tilde{L}_{1,\varphi}^{(2)} m_{>L}^{(2)}, \widehat{v}(t')\big)\big]_0^t \\
&= \int_0^t \Lambda_\varphi^{(3)}\big(\sum_{j=4}^{9}\tilde{M}_{j,\varphi}^{(3)} + \sum_{j=2}^{3}\tilde{L}_{j,\varphi}^{(3)}\Phi_\varphi^{(3)} m_{>L}^{(3)} + \sum_{j=2}^{3}\tilde{L}_{j,\varphi}^{(3)}\Phi_\varphi^{(3)} m_{\le L}^{(3)}, \widehat{v}(t')\big)\,dt' \\
&+ \int_0^t \Lambda_\varphi^{(4)}\big([[2\tilde{L}_{1,\varphi}^{(2)} m_{>L}^{(2)}]_{ext1}^{(4)}[-Q_2^{(3)}]_{ext2}^{(4)}]_{sym}^{(4)}, \widehat{v}(t')\big)\,dt'.
\end{aligned}$$

Following (3.5) and Proposition 2.5 with $m^{(2)} = \tilde{L}_{2,\varphi}^{(2)} m_{>L}^{(2)}$ and $N = 2$, we have

$$\begin{aligned}
&\int_0^t \Lambda_\varphi^{(2)}\big(\tilde{L}_{2,\varphi}^{(2)}\Phi_\varphi^{(2)} m_{>L}^{(2)}, \widehat{v}(t')\big)\,dt' + \big[\Lambda_\varphi^{(2)}\big(\tilde{L}_{2,\varphi}^{(2)} m_{>L}^{(2)}, \widehat{v}(t')\big)\big]_0^t \\
&= \int_0^t \Lambda_\varphi^{(3)}\big(\tilde{M}_{10,\varphi}^{(3)}, \widehat{v}(t')\big) + \Lambda_\varphi^{(4)}\big([[2\tilde{L}_{2,\varphi}^{(2)} m_{>L}^{(2)}]_{ext1}^{(4)}[-Q_2^{(3)}]_{ext2}^{(4)}]_{sym}^{(4)}, \widehat{v}(t')\big)\,dt'.
\end{aligned} \tag{3.6}$$

From (3.5) and Proposition 2.5 with $m^{(3)} = (\tilde{L}_{1,\varphi}^{(3)} + \tilde{L}_{3,\varphi}^{(3)}) m_{>L}^{(3)}$ and $N = 3$, we have

$$\begin{aligned}
&\int_0^t \Lambda_\varphi^{(3)}\big((\tilde{L}_{1,\varphi}^{(3)} + \tilde{L}_{3,\varphi}^{(3)})\Phi_\varphi^{(3)} m_{>L}^{(3)}, \widehat{v}(t')\big)\,dt' + \big[\Lambda_\varphi^{(3)}\big((\tilde{L}_{1,\varphi}^{(3)} + \tilde{L}_{3,\varphi}^{(3)}) m_{>L}^{(3)}, \widehat{v}(t')\big)\big]_0^t \\
&= \int_0^t \Lambda_\varphi^{(4)}(\tilde{M}_{2,\varphi}^{(4)}, \widehat{v}(t')\big) + \Lambda_\varphi^{(5)}\big([[3\tilde{L}_{1,\varphi}^{(3)} m_{>L}^{(3)}]_{ext1}^{(5)}[-Q_2^{(3)}]_{ext2}^{(5)}]_{sym}^{(5)}, \widehat{v}(t')\big)\,dt'.
\end{aligned} \tag{3.7}$$

Following (3.5) and Proposition 2.5 with $m^{(3)} = \tilde{L}_{2,\varphi}^{(3)} m_{>L}^{(3)}$ and $N = 3$, we have

$$\begin{aligned}
&\int_0^t \Lambda_\varphi^{(3)}\big(\tilde{L}_{2,\varphi}^{(3)}\Phi_\varphi^{(3)} m_{>L}^{(3)}, \widehat{v}(t')\big)\,dt' + \big[\Lambda_\varphi^{(3)}\big(\tilde{L}_{2,\varphi}^{(3)} m_{>L}^{(3)}, \widehat{v}(t')\big)\big]_0^t \\
&= \int_0^t \Lambda_\varphi^{(4)}\big(\sum_{j=3}^{8}\tilde{M}_{j,\varphi}^{(4)} + \tilde{L}_{1,\varphi}^{(4)}\Phi_\varphi^{(4)} m_{>L}^{(4)} + \tilde{L}_{1,\varphi}^{(4)}\Phi_\varphi^{(4)} m_{\le L}^{(4)}, \widehat{v}(t')\big)\,dt' \\
&+ \int_0^t \Lambda_\varphi^{(5)}\big([[3\tilde{L}_{2,\varphi}^{(3)} m_{>L}^{(3)}]_{ext1}^{(5)}[-Q_2^{(3)}]_{ext2}^{(5)}]_{sym}^{(5)}, \widehat{v}(t')\big)\,dt'.
\end{aligned} \tag{3.8}$$

From (3.5) and Proposition 2.5 with $m^{(4)} = \tilde{L}_{1,\varphi}^{(4)} m_{>L}^{(4)}$ and $N = 4$, we have

$$\begin{aligned}
&\int_0^t \Lambda_\varphi^{(4)}\big(\tilde{L}_{1,\varphi}^{(4)}\Phi_\varphi^{(4)} m_{>L}^{(4)}, \widehat{v}(t')\big)\,dt' + \big[\Lambda_\varphi^{(4)}\big(\tilde{L}_{3,\varphi}^{(4)} m_{>L}^{(4)}, \widehat{v}(t')\big)\big]_0^t \\
&= \int_0^t \Lambda_\varphi^{(5)}\big(\tilde{M}_{2,\varphi}^{(5)}, \widehat{v}(t')\big)\,dt' + \int_0^t \Lambda_\varphi^{(6)}\big(\tilde{M}_{1,\varphi}^{(6)}, \widehat{v}(t')\big)\,dt'.
\end{aligned} \tag{3.9}$$

Collecting (3.2)–(3.4) and (3.6)–(3.9), we obtain (3.1).

□

## § 4. multilinear estimates

In this section, we establish a variant of the Sobolev embedding theorems to estimate the nonlinear terms in (3.1).

We define the dyadic projection $\{P_{M_j}\}$ as

$$P_{M_j} v = \mathcal{F}_k^{-1}\left[\chi_{\{|k|\sim M_j\}} \hat{v}(k)\right]$$

where $\{M_j\}$ are dyadic numbers. For simplicity, $v_{M_j}$ denotes $P_{M_j} v$.

We present the multilinear estimates controlling the terms in Proposition 3.1. Without loss of generality, we assume that $\hat{v}$ is nonnegative in the following.

**Lemma 4.1.** *Let $s \geq 3/2$, $N \geq 2$, $0 \leq a < 1$ and an $N$-multiplier $M^{(N)}$ satisfy*

$$\left|M^{(N)}\right| \lesssim \prod_{i=1}^{N-1} |k_i|^a m_{H1}^{(N)}. \tag{4.1}$$

*Then, it follows that*

$$\left\|\mathcal{F}_k^{-1} \sum_{\Gamma_k^{(N)}} M^{(N)} \prod_{i=1}^{N} \hat{v}(k_i)\right\|_{H^s} \lesssim \|v\|_{H^s}^N, \tag{4.2}$$

$$\left\|\mathcal{F}_k^{-1} \sum_{\Gamma_k^{(N)}} M^{(N)} \Big(\prod_{i=1}^{N} \hat{v}(k_i) - \prod_{i=1}^{N} \hat{w}(k_i)\Big)\right\|_{H^s} \lesssim \big(\|v\|_{H^s} + \|w\|_{H^s}\big)^{N-1} \|v-w\|_{H^s}. \tag{4.3}$$

*Proof.* We only (4.2) since (4.3) follows in a similar manner. By the Sobolev embedding theorem, we have

$$\begin{aligned}
&\left\|\mathcal{F}_k^{-1} \sum_{\Gamma_k^{(N)}} M^{(N)} \prod_{i=1}^{N} \hat{v}(k_i)\right\|_{H^s} \\
&\lesssim \sum_{M_i \ll M_N} \prod_{i=1}^{N-1} \|\langle \partial_x \rangle^a v_{M_i}\|_{L^\infty} \|\langle \partial_x \rangle^s v_{M_N}\|_{L^2} \\
&\lesssim \|v\|_{H^{3/2}}^{N-1} \|v\|_{H^s}.
\end{aligned}$$

□

**Lemma 4.2.** *Let $s \geq 3/2$, $N \geq 2$, $1 \leq l \leq N$ and an $N$ multiplier $M^{(N)}$ satisfy*

$$\left|M^{(N)}\right| \lesssim |k_N|^{l-1} m_{Hl}^{(N)}. \tag{4.4}$$

*Then, we have (4.2) and (4.3).*

*Proof.* A simple calculation yields that

$$\langle k_{1,2,\cdots,N}\rangle^s |k_N|^{l-1} m_{Hl}^{(N)} \lesssim \prod_{j=N-l+1}^{N} \langle k_j\rangle^{s/l+(l-1)/l} m_{Hl}^{(N)}. \tag{4.5}$$

From (4.5), we use the embedding theorem and the Hölder inequality to have

$$\begin{aligned}
&\Big\|\mathcal{F}_k^{-1} \sum_{\Gamma_k^{(N)}} M^{(N)} \prod_{i=1}^{N} \hat{v}(k_i)\Big\|_{H^s} \\
&\lesssim \sum_{M_i \ll M_j \sim M_N} \prod_{i=1}^{N-l} \|v_{M_i}\|_{L^\infty} \prod_{j=N-l+1}^{N} \|\langle \partial_x\rangle^{s/l+(l-1)/l} v_{M_j}\|_{L^{2l}} \\
&\lesssim \|v\|_{H^{1/2+}}^{N-l} \|v\|_{H^{s/l+3(l-1)/2l}}^{l},
\end{aligned}$$

which shows (4.2) since $s \geq 3/2$. In the similar manner as above, we obtain (4.3). $\square$

**Lemma 4.3.** *Let $s \geq 3/2$, $N \geq 3$ and an $N$ multiplier $M^{(N)}$ satisfy*

$$\left|M^{(N)}\right| \lesssim \langle k_{1,2,\cdots,N}\rangle^{-1} |k_N|^2 m_{H2}^{(N)}. \tag{4.6}$$

*Then, we have (4.2) and (4.3).*

*Proof.* A simple calculation yields that

$$\langle k_{1,2,\cdots,N}\rangle^{s-1} |k_{N-1}k_N| m_{H2}^{(N)} \lesssim |k_{N-1}k_N|^{s/2+1/2} m_{H2}^{(N)}. \tag{4.7}$$

From (4.7), the embedding theorem and the Hölder inequality show that

$$\begin{aligned}
&\Big\|\mathcal{F}_k^{-1} \sum_{\Gamma_k^{(N)}} M^{(N)} \prod_{i=1}^{N} \hat{v}(k_i)\Big\|_{H^s} \\
&\lesssim \|v\|_{H^{1/2+}}^{N-2} \sum_{M_{N-1} \sim M_N} \|\langle \partial_x\rangle^{s/2+1/2} v_{M_{N-1}}\|_{L^4} \big\| \langle \partial_x\rangle^{s/2+1/2} v_{M_N}\big\|_{L^4} \\
&\lesssim \|v\|_{H^{1/2+}}^{N-2} \|v\|_{H^{s/2+3/4}}^{2},
\end{aligned}$$

which implies (4.2) for $s \geq 3/2$. Similarly, we have (4.3). $\square$

## § 5. pointwise upper bounds

In this section, we give pointwise upper bounds on the multipliers defined in Proposition 3.1.

Combing the lemmas constructed in the previous section, we obtain the appropriate bounds for the nonlinear terms in (3.1). Here we set

$$\begin{aligned}
I_1 &= \big\{(2,1),(2,2),(3,1),(3,2),(3,3),(4,1)\big\},\\
I_2 &= \big\{(3,2),(3,7),(4,6\big\},\\
I_3 &= \big\{(3,l_1),(4,l_2),(5,1),(5,2),(6,1)\,|\,l_1=1,\cdots,10,\ l_2=1,\cdots,8\big\}\setminus I_2.
\end{aligned}$$

**Lemma 5.1.** *Let $r>0$, $L \gtrsim r$ and $f \in B_r(L^2(\mathbb{T}))$. Then there exists $a>0$ such that*

$$\big|\tilde{L}^{(N)}_{j,f} m^{(N)}_{>L}\big| \lesssim L^{-a}. \tag{5.1}$$

*for $(N,j) \in I_1$.*

Before we describe the proof of Lemma 5.1, we remark that

$$\big|\big[M^{(N)}\big]^{(N)}_{sym}\big| \lesssim \big[\big|M^{(N)}\big|\big]^{(3)}_{sym}.$$

for any $N$-multiplier $M^{(N)}$. Thus it suffices to show the appropriate bounds for $L^{(N)}_{j,f}$ and $M^{(N)}_{j,f}$ in Lemma 5.1 (and Lemma 5.2 below).

*Proof of Lemma 5.1.* From Lemmas 2.1 and 2.2 , we obtain

$$\big|L^{(2)}_{1,f} m^{(2)}_{>L}\big| \lesssim \frac{1}{|k_1 k_2|}\, m^{(2)}_{H1} m^{(2)}_{NR} m^{(2)}_{>L}, \tag{5.2}$$

$$\big|L^{(2)}_{2,f} m^{(2)}_{>L}\big| \lesssim \frac{1}{|k_2|^2}\, m^{(2)}_{H2} m^{(2)}_{NR} m^{(2)}_{>L}, \tag{5.3}$$

$$\big|L^{(3)}_{1,f} m^{(3)}_{>L}\big| \lesssim \frac{1}{|k_{1,2}||k_3|^3}\, m^{(3)}_{H1} m^{(3)}_{NR} m^{(3)}_{>L}, \tag{5.4}$$

$$\big|L^{(3)}_{2,f} m^{(3)}_{>L}\big| \lesssim \frac{1}{|k_1 k_2||k_3|^2}\, m^{(3)}_{H1}(1-m^{(3)}_{R1})(1-m^{(3)}_{R2}) m^{(3)}_{>L}, \tag{5.5}$$

$$\big|L^{(3)}_{3,f} m^{(3)}_{>L}\big| \lesssim \frac{1}{|k_1||k_{2,3}||k_3|^2}\, m^{(3)}_{H2} m^{(3)}_{NR}(1-m^{(3)}_{R2}) m^{(3)}_{>L}. \tag{5.6}$$

From Lemma 2.3, we have

$$\big|\Phi^{(4)}_f\big| m^{(4)}_{H1}(1-m^{(4)}_{R1})(1-m^{(4)}_{R5}) m^{(4)}_{>L} \gtrsim |k_{1,2,3}||k_4|^4 m^{(4)}_{H1}(1-m^{(4)}_{R1})(1-m^{(4)}_{R5}) m^{(4)}_{>L}.$$

Thus, it follows that

$$\big|L^{(4)}_{1,f} m^{(4)}_{>L}\big| \lesssim \frac{1}{|k_1 k_2 k_{1,2,3}||k_4|^3} m^{(4)}_{H1}(1-m^{(4)}_{R1})(1-m^{(4)}_{R2})(1-m^{(4)}_{R5}) m^{(4)}_{>L}. \tag{5.7}$$

□

**Lemma 5.2.** *Let $r>0$, $L\gtrsim r$ and $f\in B_r(L^2(\mathbb{T}))$. Then $\tilde{M}_{j,f}^{(N)}$ with $(N,j)\in I_3$ satisfies the condition (4.1), (4.4) or (4.6).*

From (5.5), we have

$$|M_{7,f}^{(4)}|\lesssim\frac{|k_4|}{|k_1k_2|}m_{H1}^{(4)}m_{R5}^{(4)}(1-m_{R2}^{(4)})\lesssim|k_2k_3|^{1/8}m_{H1}^{(4)}.$$

Thus $M_{7,f}^{(4)}$ satisfies (4.1). In this way, Lemma 5.2 follows by Lemma 5.1. For the details, see the original paper [8].

**Lemma 5.3.** *Let $r>0$, $L\gtrsim r$ and $f,g\in B_r(L^2(\mathbb{T}))$. Then there exists $C>0$ such that*

$$\big|(\tilde{L}_{j,f}^{(N)}-\tilde{L}_{j,g}^{(N)})m_{>L}^{(N)}\big|\le C\frac{|E_0(f)-E_0(g)|}{|k_N|}|\tilde{L}_{j,g}^{(N)}| \tag{5.8}$$

*for $(N,j)\in I_1$ and*

$$\big|\tilde{M}_{j,f}^{(N)}-\tilde{M}_{j,g}^{(N)}\big|\le C|E_0(f)-E_0(g)||\tilde{M}_{j,g}^{(N)}| \tag{5.9}$$

*for $(N,j)\in I_3$.*

*Proof.* From Lemmas 2.1–2.3, we obtain (5.8) immediately. Following (5.8) and the definition in Proposition 3.1, we have (5.9). □

We will show that the resonant parts corresponding to $\tilde{M}_{2,\varphi}^{(3)}+\tilde{M}_{7,\varphi}^{(3)}$ and $\tilde{M}_{6,\varphi}^{(4)}$ are canceled out. Put

$$m_{HR}^{(3)}=m_{H1}^{(3)}m_{R1}^{(3)}(1-m_{R2}^{(3)}),\quad m_{HR}^{(4)}=m_{H1}^{(4)}m_{R1}^{(4)}(1-m_{R2}^{(4)}).$$

The following proposition implies that the resonant parts with a loss of derivatives arising from $\Lambda_\varphi^{(2)}(-Q_2^{(2)}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)},\hat{v})$ by the normal form reduction can be canceled by the resonant parts corresponding to $\tilde{M}_{2,\varphi}^{(3)}$.

**Proposition 5.4.** *It follows that*

$$\tilde{M}_{2,\varphi}^{(3)}+\tilde{M}_{7,\varphi}^{(3)}=0. \tag{5.10}$$

*Proof.* From (2.1), we have

$$
\begin{aligned}
M_{7,\varphi}^{(3)} &= \big[-\frac{\beta}{2}\frac{q_1^{(2)}}{\Phi_0^{(2)}}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext1}^{(3)}\big[-\frac{\beta}{4}q_1^{(2)}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(3)}2m_{H1}^{(3)}m_{R1}^{(3)} \\
&= \frac{\beta^2}{4}\big[\frac{q_1^{(2)}}{\Phi_0^{(2)}}m_{NR}^{(2)}\big]_{ext1}^{(3)}\big[q_1^{(2)}m_{NR}^{(2)}\big]_{ext2}^{(3)}m_{H1}^{(3)}m_{R1}^{(3)} \\
&= \frac{\beta^2}{4}\big[\frac{q_1^{(2)}}{\Phi_0^{(2)}}\big]_{ext1}^{(3)}\big[q_1^{(2)}\big]_{ext2}^{(3)}m_{HR}^{(3)} = -\frac{\beta^2}{5}i\big[\frac{1}{k_1k_2}\big]_{ext1}^{(3)}\big[k_{1,2}(k_1^2+k_1k_2+k_2^2)\big]_{ext2}^{(3)}m_{HR}^{(3)} \\
&= -\frac{\beta^2}{5}i\frac{k_2^2+k_2k_3+k_3^2}{k_1}m_{HR}^{(3)} = -\frac{\beta^2}{5}i\big(k_1-k_3+\frac{k_3^2}{k_1}\big)m_{HR}^{(3)}
\end{aligned}
$$

Here we used $k_1+k_2=0$ when $m_{HR}^{(3)}\neq 0$. Thus, we have

$$
\begin{aligned}
M_{1,\varphi}^{(3)}+M_{7,\varphi}^{(3)} &= -\frac{\beta^2}{5}i\big(k_1-k_3+\frac{k_3^2}{k_1}\big)m_{HR}^{(3)} - \frac{\beta^2}{5}ik_{1,2,3}m_{HR}^{(3)} \\
&= -\frac{\beta^2}{5}i\big(k_1+\frac{k_3^2}{k_1}\big)m_{HR}^{(3)}
\end{aligned}
$$

We notice that $m_{HR}^{(3)}(k_1,k_2,k_3)=m_{HR}^{(3)}(k_2,k_1,k_3)$. By this symmetry, we obtain

$$
\begin{aligned}
\tilde{M}_{1,\varphi}^{(3)}+\tilde{M}_{7,\varphi}^{(3)} &= -\frac{\beta^2}{10}i\big[\big(k_1+\frac{k_3^2}{k_1}\big)m_{HR}^{(3)}+\big(k_2+\frac{k_3}{k_2}\big)m_{HR}^{(3)}\big]_{sym}^{(3)} \\
&= -\frac{\beta^2}{10}i\big[(k_1+k_2)m_{HR}^{(3)}\big]_{sym}^{(3)} - \frac{\beta^2}{10}i\big[\frac{k_1+k_2}{k_1k_2}k_3^2\big]_{sym}^{(3)} = 0.
\end{aligned}
$$

□

It seems that a loss of derivatives exists in the resonant parts corresponding to $\tilde{M}_{6,\varphi}^{(4)}$. However, the following proposition claims that derivative losses are excluded in this resonant parts thanks to the algebraic structure.

**Proposition 5.5.** *It follows that*

$$
|\tilde{M}_{6,\varphi}^{(4)}| \lesssim 1. \tag{5.11}
$$

*Proof.* We put $m_{HNR}^{(3)}=m_{H1}^{(3)}(1-m_{R1}^{(3)})(1-m_{R2}^{(3)})$. Then, we have $m_{HNR}^{(3)}(k_1,k_2,k_3)=m_{HNR}^{(3)}(k_2,k_1,k_3)$. We use this property to obtain

$$
\begin{aligned}
\tilde{L}_{2,0}^{(3)}\Phi_0^{(3)} =& \big[-\frac{\beta^2}{10}i\big(\frac{k_1+k_2}{k_1k_2}k_3^2+\frac{k_1^2+k_2^2}{k_1k_2}k_3+\frac{k_1^3+k_3^3}{k_1k_2}\big)m_{NHR}^{(3)}\big]_{sym}^{(3)} \\
=:& \tilde{I}_1^{(3)}\Phi_0^{(3)}+\tilde{I}_2^{(3)}\Phi_0^{(3)}
\end{aligned}
$$

where

$$I_1^{(3)}\Phi_0^{(3)} = -\frac{\beta^2}{10} i \frac{k_1+k_2}{k_1k_2} k_3^2 m_{NHR}^{(3)},$$
$$I_2^{(3)}\Phi_0^{(3)} = \tilde{L}_{2,0}^{(3)}\Phi_0^{(3)} - \tilde{I}_1^{(3)}\Phi_0^{(3)}.$$

We set

$$J_1^{(4)} := \big[3\tilde{I}_1^{(3)}\big]_{ext1}^{(4)}\big[-Q_1^{(2)}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(4)}2m_{H1}^{(4)}m_{R1}^{(4)},$$
$$J_2^{(4)} := \big[3\tilde{I}_2^{(3)}\big]_{ext1}^{(4)}\big[-Q_1^{(2)}m_{NR}^{(2)}[2m_{H1}^{(2)}]_{sym}^{(2)}\big]_{ext2}^{(4)}2m_{H1}^{(4)}m_{R1}^{(4)}.$$

Obviously, $\tilde{J}_1^{(4)} + \tilde{J}_2^{(4)} = \tilde{M}_{9,\varphi}^{(4)}$. From

$$|\Phi_0^{(3)}|m_{HNR}^{(3)} \gtrsim |k_{1,2}||k_3|^4 m_{HNR}^{(3)} \tag{5.12}$$

by (2.2), $|J_2^{(4)}|$ is bounded. Clearly, it follows that

$$|[J_2^{(4)}]_{sym}^{(4)}| \lesssim [|J_2^{(4)}|]_{sym}^{(4)} \lesssim 1. \tag{5.13}$$

We use (2.2) to have

$$J_1^{(4)} = \big[-\frac{3\beta^2}{10} i \frac{k_1+k_2}{k_1k_2}\frac{k_3^2}{\Phi_0^{(3)}} m_{NR}^{(3)}(1-m_{R2}^{(3)})\big]_{ext1}^{(4)}\big[\frac{\beta}{2} i k_{1,2}(k_1^2+k_1k_2+k_2^2)m_{NR}^{(2)}\big]_{ext2}^{(4)}2m_{H1}^{(4)}m_{R1}^{(4)}$$
$$= \frac{3\beta^3}{10}\frac{k_1+k_2}{k_1k_2}\frac{(k_4-k_{1,2})^3}{[\Phi_0^{(3)}]_{ext1}^{(4)}}(k_4^2+k_3k_4+k_3^2)m_{HR}^{(4)} =: J_{1,1}^{(4)} + J_{1,2}^{(4)}$$

where

$$J_{1,1}^{(4)} := \frac{3\beta^3}{10}\frac{k_1+k_2}{k_1k_2}\frac{(k_4-k_{1,2})^3}{[\Phi_0^{(3)}]_{ext1}^{(4)}}k_4^2 m_{HR}^{(4)},$$
$$J_{1,2}^{(4)} := J_1^{(4)} - J_{1,1}^{(4)}.$$

We apply (5.12) to obtain

$$|[J_{1,2}^{(4)}]_{sym}^{(4)}| \lesssim |[J_{1,2}^{(4)}]_{sym}^{(4)}| \lesssim 1. \tag{5.14}$$

It suffice to show the multiplier $|\tilde{J}_{1,1}^{(4)}|$ is bounded. Here we set

$$S_{1,2}^{(4)} = -i[k_{1,2,3,4}^5 - \{k_1^5 + k_2^5 + (k_4-k_{1,2})^5\}],$$
$$S_{2,3}^{(4)} = -i[k_{1,2,3,4}^5 - \{k_2^5 + k_3^5 + (k_4-k_{2,3})^5\}],$$
$$S_{3,1}^{(4)} = -i[k_{1,2,3,4}^5 - \{k_1^5 + k_3^5 + (k_4-k_{1,3})^5\}].$$

We introduce the multiplier $K_{1,1}^{(4)}$ as

$$K_{1,1}^{(4)} = \frac{\beta^3}{10} k_4^2 m_{HR}^{(4)} \Big( \frac{k_{1,2}(k_4 - k_{1,2})^3}{k_1 k_2 S_{1,2}^{(4)}} + \frac{k_{2,3}(k_4 - k_{2,3})^3}{k_2 k_3 S_{2,3}^{(4)}} + \frac{k_{1,3}(k_3 - k_{1,3})^3}{k_1 k_3 S_{1,3}^{(4)}} \Big).$$

We notice that $[K_{1,1}^{(4)}]_{sym}^{(4)} = [J_{1,1}^{(4)}]_{sym}^{(4)}$ since

$$m_{HR}^{(4)}(k_1, k_2, k_3, k_4) = m_{HR}^{(4)}(k_2, k_3, k_1, k_4) = m_{HR}^{(4)}(k_3, k_1, k_2, k_4).$$

A direct computation yields that there exists a sequence of polynomial functions $\{f_j(k_1, k_2, k_3)\}_{j=0}^{11}$ such that

$$\begin{aligned} K_{1,1}^{(4)} = \frac{\beta^3}{10} & \frac{k_4^2 \; m_{HR}^{(4)}}{k_1 k_2 k_3 S_{1,2}^{(4)} S_{2,3}^{(4)} S_{1,3}^{(4)}} \\ & \times \{ f_0(k_1, k_2, k_3) k_4^{11} + f_1(k_1, k_2, k_3) k_4^{10} + \cdots + f_{11}(k_1, k_2, k_3) \}. \end{aligned}$$

where $f_j(k_1, k_2, k_3)$ satisfy

$$f_0(k_1, k_2, k_3) m_{HR}^{(4)} = 25 k_{1,2,3} k_1 k_2 k_3 m_{HR}^{(4)} = 0$$

and

$$|f_j(k_1, k_2, k_3) k_4^{11-j}| m_{HR}^{(4)} \lesssim |k_1 k_2 k_3| \max_{j=1,2,3} \{|k_j|^2\} |k_4|^{10} m_{HR}^{(4)}$$

for $1 \leq j \leq 11$. Following

$$|k_1 k_2 k_3 S_{1,2}^{(4)} S_{2,3}^{(4)} S_{1,3}^{(4)}| m_{HR}^{(4)} \gtrsim |k_1^2 k_2^2 k_3^2 k_4^{12}| m_{HR}^{(4)},$$

we have $|K_{1,1}^{(4)}| \lesssim 1$. From $[K_{1,1}^{(4)}]_{sym}^{(4)} = [J_{1,1}^{(4)}]_{sym}^{(4)}$, it follows that

$$|[J_{1,1}^{(4)}]_{sym}^{(4)}| = |[K_{1,1}^{(4)}]_{sym}^{(4)}| \lesssim [|K_{1,1}^{(4)}|]_{sym}^{(4)} \lesssim 1. \tag{5.15}$$

Collecting (5.13), (5.14) and (5.15), we obtain (5.11). □

Propositions 5.4 and 5.5 shows that the resonant parts with derivative losses are excluded from the nonlinear terms $F_{\varphi,L}$ and $G_{\varphi,L}$.

## § 6. Proof of the main theorem

In this section, we prove the main theorem by the contraction argument.

We apply a variant of the Sobolev embedding theorems Lemmas 4.1–4.3 to the pointwise upper bounds obtained in Lemmas 5.1 and 5.2 to have the following lemma.

**Lemma 6.1.** *Let $s \geq 3/2$ and $r > 0$. Then there exist $C > 0$ and $a, b > 0$ such that the following estimates hold for any $L \gtrsim r$, $v, w \in H^s(\mathbb{T})$ and $f \in B_r(L^2(\mathbb{T}))$:*

$$\begin{aligned} &\big\|\mathcal{F}_k^{-1}\big[\Lambda_f^{(N)}\big(\tilde{L}_{j,f}^{(N)} m_{>L}^{(N)}, \hat{v}\big)\big]\big\|_{H^s} \leq CL^{-a}\|v\|_{H^s}^N, \\ &\big\|\mathcal{F}_k^{-1}\big[\Lambda_f^{(N)}\big(\tilde{L}_{j,f}^{(N)} m_{>L}^{(N)}, \hat{v}\big) - \Lambda_f^{(N)}\big(\tilde{L}_{j,f}^{(N)} m_{>L}^{(N)}, \hat{w}\big)\big]\|_{H^s} \\ &\qquad \leq CL^{-a}(\|v\|_{H^s} + \|w\|_{H^s})^{N-1}\|v - w\|_{H^s} \end{aligned} \tag{6.1}$$

*where $(N, j) \in I_1$ and*

$$\begin{aligned} &\big\|\mathcal{F}_k^{-1}\big[\Lambda_f^{(N)}\big(\tilde{M}_{j,f}^{(N)}, \hat{v}\big)\big]\big\|_{H^s} \leq CL^b\|v\|_{H^s}^N, \\ &\big\|\mathcal{F}_k^{-1}\big[\Lambda_f^{(N)}\big(\tilde{M}_{j,f}^{(N)}, \hat{v}\big) - \Lambda_f^{(N)}\big(\tilde{M}_{j,f}^{(N)}, \hat{w}\big)\big]\big\|_{H^s} \\ &\qquad \leq CL^b(\|v\|_{H^s} + \|w\|_{H^s})^{N-1}\|v - w\|_{H^s} \end{aligned} \tag{6.2}$$

*where $(N, j) \in I_3$.*

*Remark* 6.1. We remark that $\tilde{M}_{2,\varphi}^{(3)}$, $\tilde{M}_{7,\varphi}^{(3)}$ and $\tilde{M}_{6,\varphi}^{(4)}$ do not depend on initial data $\varphi$. From Propositions 5.4 and 5.5, we have the following for any $v \in H^s(\mathbb{T})$, $f, g \in L^2(\mathbb{T})$:

$$\Lambda_f^{(3)}\big(\tilde{M}_{2,g}^{(3)} + \tilde{M}_{7,\varphi}^{(3)}, \hat{v}\big) = 0 \quad \text{and} \quad \big|\Lambda_f^{(4)}\big(\tilde{M}_{6,g}^{(4)}, \tilde{v}\big)\big| \lesssim \big|\Lambda_f^{(4)}\big(1, \hat{v}\big)\big|.$$

**Proposition 6.2.** *Let $s \geq 3/2$ and $r > 0$. Then, there exist $C > 0$ and $a, b > 0$ such that the following estimates hold for any $L \gtrsim r$, $v, w \in H^s(\mathbb{T})$ and $f, g \in B_r(L^2(\mathbb{T}))$:*

$$\big\|F_{f,L}(v) - F_{g,L}(w)\|_{H^s} \leq L^{-a}(1 + \|v\|_{H^s} + \|w\|_{H^s})^3\big(C\|v - w\|_{H^s} + C_1\|v\|_{H^s}\big), \tag{6.3}$$

$$\big\|G_{f,L}(v) - G_{g,L}(w)\big\|_{H^s} \leq L^b(1 + \|v\|_{H^s} + \|w\|_{H^s})^5\big(C\|v - w\|_{H^s} + C_1\|v\|_{H^s}\big) \tag{6.4}$$

*where $C_1$ is a constant depending only on $|E_0(f) - E_0(g)|$, $C_1 \to 0$ when $E_0(f) \to E_0(g)$ and $C_1 = 0$ when $E_0(f) = E_0(g)$.*

*Proof.* We only prove (6.4) since (6.3) follows in a similar manner. A direct calculation yields that

$$\begin{aligned} &\Lambda_f^{(N)}\big(\tilde{M}_{j,f}^{(N)}, \hat{v}\big) - \Lambda_g^{(N)}\big(\tilde{M}_{j,g}^{(N)}, \hat{w}\big) = \sum_{\Gamma_k^{(N)}} (e^{-t\Phi_f} - e^{-t\Phi_g})\tilde{M}_{j,f}^{(N)} \prod_{i=1}^N \hat{v}(k_i) \\ &+ \sum_{\Gamma_k^{(N)}} e^{-t\Phi_g}(\tilde{M}_{j,f}^{(N)} - \tilde{M}_{j,g}^{(N)}) \prod_{i=1}^N \hat{v}(k_i) + \Big(\Lambda_g^{(N)}\big(\tilde{M}_{j,g}^{(N)}, \hat{v}\big) - \Lambda_g^{(N)}\big(\tilde{M}_{j,g}^{(N)}, \hat{w}\big)\Big) \\ &=: \hat{J}_1(k) + \hat{J}_2(k) + \hat{J}_3(k) \end{aligned}$$

Here we consider the case $(N, j) \in I_3$. From Lemma 2.4 and (6.2), there exist $b > 0$ and $C_1 > 0$ such that

$$\big\|\mathcal{F}_k^{-1}\big[\hat{J}_1(k)\big]\big\|_{H^s} \leq C_1L^b\|v\|_{H^s}^N, \tag{6.5}$$

$C_1 \to 0$ when $E_0(f) \to E_0(g)$ and $C_1 = 0$ when $E_0(f) = E_0(g)$.
Following Lemma 5.3 and (6.2), we have

$$\big\|\mathcal{F}_k^{-1}\big[\hat{J}_2(k)\big]\big\|_{H^s} \le CL^b |E_0(f) - E_0(g)| \|v\|_{H^s}^N. \tag{6.6}$$

(6.2) shows that

$$\big\|\mathcal{F}_k^{-1}\big[\hat{J}_3(k)\big]\big\|_{H^s} \le CL^b (\|v\|_{H^s} + \|w\|_{H^s})^{N-1} \|v - w\|_{H^s}. \tag{6.7}$$

Collecting (6.5)–(6.7), we have

$$\begin{aligned} &\big\|\mathcal{F}_k^{-1}\big[\Lambda_f^{(N)}\big(\tilde{M}_{j,f}^{(N)}, \hat{v}\big) - \Lambda_g^{(N)}\big(\tilde{M}_{j,g}^{(N)}, \hat{v}\big)\big]\big\|_{H^s} \\ &\quad \le CL^b (\|v\|_{H^s} + \|w\|_{H^s})^{N-1} \big(C\|v - w\|_{H^s} + C_1 \|v\|_{H^s}\big) \end{aligned} \tag{6.8}$$

for $(N, j) \in I_3$. Following (6.8) and Remark 6.1, we obtain (6.4). □

From the above proposition, we immediately obtain the following corollary.

**Corollary 6.3.** *Let $s \ge 3/2$ and $r > 0$. Then, there exist $C > 0$ and $1 > \theta > 0$ such that the following estimate holds for any $1 \ge T > 0$, $\varphi_1, \varphi_2 \in B_r(H^s(\mathbb{T}))$ and any solution $u_1 \in C([-T, T] : H^s(\mathbb{T}))$ (resp. $u_2 \in C([-T, T] : H^s(\mathbb{T}))$) to (1.6) with initial data $\varphi_1$ (resp. $\varphi_2$):*

$$\begin{aligned} \|u_1 - u_2\|_{L_T^\infty H^s} \le &\|\varphi_1 - \varphi_2\|_{H^s} + T^\theta (1 + \|u_1\|_{L_T^\infty H^{3/2}} + \|u_2\|_{L_T^\infty H^{3/2}})^5 \\ &\times (C\|u_1 - u_2\|_{L_T^\infty H^s} + C_1 \|u_1\|_{L_T^\infty H^s}), \end{aligned} \tag{6.9}$$

*where $C_1$ is a constant depending only on $|E_0(\varphi_1) - E_0(\varphi_2)|$, $C_1 \to 0$ when $E_0(\varphi_1) \to E_0(\varphi_2)$ and $C_1 = 0$ when $E_0(\varphi_1) = E_0(\varphi_2)$*

By the standard argument and Corollary 6.3, we can prove the local well-posedness of (3.1). However, it is not clear whether the solution to (3.1) satisfies (1.6) or not. To overcome this difficulty, we use the following proposition.

**Proposition 6.4.** *Let $m \in N$ be sufficient large. Then, (1.1)–(1.2) is locally well-posed in $H^m(\mathbb{T})$ on $[-T, T]$ without any condition $\alpha$, $\beta$ and $\gamma$. The existence time $T$ depends on only $\|\varphi\|_{H^m}$.*

This result is a part of the paper of fifth order dispersive equations by Tsugawa [17]. His proof is based on the modified energy method developed by S. Kwon [14].

Following Corollary 6.3 and Proposition 6.4, we obtain the following result.

**Proposition 6.5.** *Let $s \ge 3/2$ and $\beta = 2\alpha$. Then, for any $\varphi \in H^s(\mathbb{T})$, there exists $T = T(\|\varphi\|_{H^{3/2}}) > 0$ such that there exists a unique solution $u \in C([-T, T] : H^s(\mathbb{T}))$ to (1.6). Moreover the solution map, $H^s(\mathbb{T}) \ni \varphi \mapsto u \in C([-T, T] : H^s(\mathbb{T}))$, is continuous.*

We omit the details of the proof of this proposition. For the details, refer the original paper [8].

Note that $E_0(u)$ and $E_1(u)$ are conserved in the rigorous sense when $u \in C([-T, T] : H^{3/2}(\mathbb{T}))$ satisfies (1.1) with $\beta = 2\alpha$. Therefore, we obtain the local well-posedness of (1.1)–(1.2) in the sense of Proposition 6.5.

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